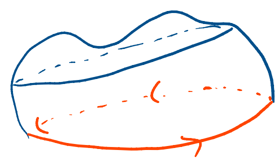
Math 53, Discussion 105

Friday, April 24, 2020 9:37 AM

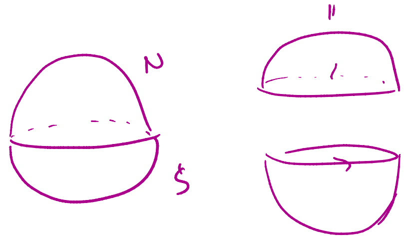
16.8 (~~Stoke's~~ Stokes' Theorem)

16.9 (Divergence Theorem)



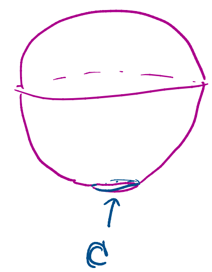
Path int. of F along C

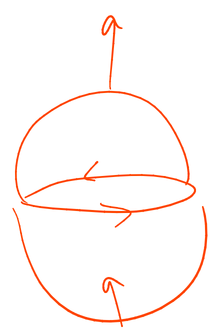
Vector field $\mathbb{R}^3 \to \mathbb{R}^3$



$$\iint \nabla\times F = \int_C F\cdot dr$$

$$= -\int_C F\cdot dr$$





Theorem (Stokes)

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S \overbrace{\nabla\times\boldsymbol{F}}^{\text{curl of }F}\cdot \overbrace{d\boldsymbol{S}}^{\text{surface integral of a v.f. (flux of }F)}$$

flux of curl of F through S

Curve positive orientation (closed)

line int. v.f.

Surface w/ boundary C

$$\iint_S (\nabla\times F)\cdot dS = \iiint \underbrace{\nabla\cdot(\nabla\times F)}_{\underbrace{\nabla\cdot\nabla\times}_{0}}\, dV$$

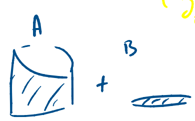
Example: compute $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ with $F = \langle -y^2, x, z^2\rangle$ and C is the intersection of the plane $y+z=2$ and $x^2+y^2=1$

$$\int_C F\cdot dr = \iint_S \nabla\times F\cdot dS$$

$$\nabla\times F = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\times\begin{pmatrix}-y^2\\ x\\ z^2\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1+2y\end{pmatrix}$$

$S =$ A + B

integral = 0



$$\iint_A \underbrace{\nabla\times F\cdot dS}_{0} = 0$$ b/c dS points perpendicular to $\nabla\times F$

dS point normal to surface

$\nabla\times F$ points in $\hat{z}$ direction

$$\iint_S G\cdot dS = \iint G\cdot n\,\|r_u\times r_v\|$$

$$\iint_B \nabla\times F\cdot dS =$$

$$= \iint_B \begin{pmatrix}0\\0\\1+2y\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix}\|r_u\times r_v\|\,dA$$

$$+\int_0^1\int_0^{2\pi}(1+2(u\sin v))u\,dv\,du$$

$r = \langle u\cos(v), u\sin v, 0\rangle$

$0\le u\le 1$

$0\le v\le 2\pi$

$$r_u\times r_v = \begin{pmatrix}\cos(v)\\ \sin(v)\\ 0\end{pmatrix}\times\begin{pmatrix}-u\sin(v)\\ u\cos(v)\\ 0\end{pmatrix}$$

Show this reduces to Green's theorem

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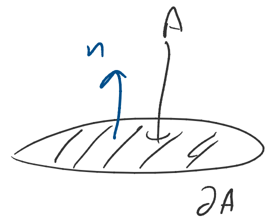
$$\iint_A \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_{\partial A} P\,dx + Q\,dy$$



$$\int_{\partial A} P\,dx + Q\,dy = \int_{\partial A} F \cdot dr \qquad \text{with } F = \langle P, Q, 0 \rangle$$

By Stokes

$$= \iint_A \nabla \times F \cdot dS = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\nabla \times F = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ \partial_x Q - \partial_y P \end{bmatrix}$$

$$(\nabla \times F) \cdot n = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$dS = n\,dS = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA$$

Interpret this to give meaning of curl

Theorem (Divergence)

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \nabla \cdot \boldsymbol{F} \, dV$$

Example: compute the flux of $F = \langle z, y, x \rangle$ over the unit sphere.

Exercises:

1) Compute $\iint_S \nabla \times \boldsymbol{F} \cdot d\boldsymbol{S}$ **for** $F = \langle 2y\cos z, e^x \sin z, xe^y \rangle$, on the upper half of the sphere centered at the origin with radius 9
2) Compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for $F = \langle yz, 2xz, e^{xy} \rangle$ C the circle $x^2 + y^2 = 16, z = 5$
3) Compute $\iint_S F \cdot dS$ for $F = \langle exy^2, xe^z, z^3 \rangle$ over the surface bounded by $y^2 + z^2 = 1$ and the $x = -1, x = 2$
4)